Binary Logistic Regression

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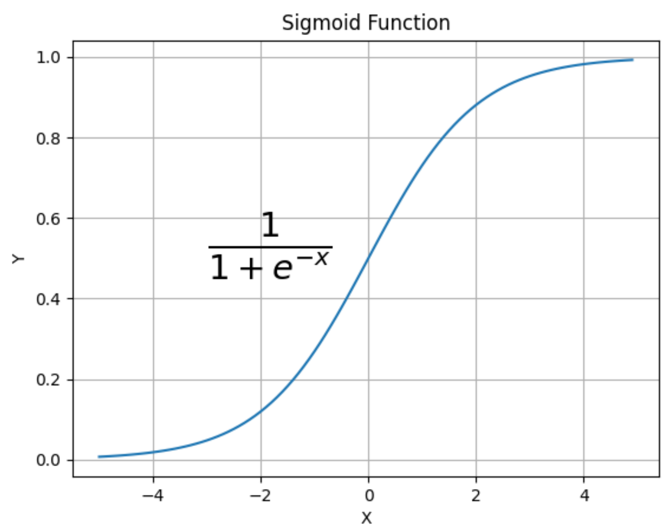
Binary logistic regression are used to predict an binary issue (win/loss, true/false) according to various parameters. First, we have to choose a polynomial function $h_w(x)$ according to the data complexity (see *data/binary_logistic.csv*). In our case, we want to predict our issue (1 or 0) according to two parameters. Thus:

$$h_w(x_1, x_2) = w_1 + w_2 x_1 + w_3 x_2 \tag{1}$$

However, the function we are looking for should return a **binary** result! To achieve this goal, we can use a sigmoid (or logistic) with the following property $\mathbb{R} \rightarrow]0;1[$ with the following form:



To this end, we can define the following function:

$$g_w(x_1, x_2) = \frac{1}{1 + e^{-h_w(x_1, x_2)}} \tag{2}$$

The next step is to define a cost function. A common approach in binary logistic function is to use the **Cross-Entropy** loss function. It is much more convenient than the classical Mean Square Error used in polynomial regression. Indeed, the gradient is stronger even for small error (see here for more informations). Thus, it looks like the following:

$$J(w) = -\frac{1}{n} \sum_{i=0}^{n} \left[y^{(i)} log(g_w(x_1^{(i)}, x_2^{(i)})) + (1 - y^{(i)}) log(1 - g_w(x_1^{(i)}, x_2^{(i)})) \right] \tag{3}$$

With n the number of observations, $x_j^{(i)}$ is the value of the j^{th} independant variable associated with the observation $y^{(i)}$. The next step is to $min_w J(w)$ for each weight w_i (performing the gradient decent, see here). Thus we compute each partial derivatives:

$$\frac{\partial J(w)}{\partial w_1} = \frac{\partial J(w)}{\partial g_w(x_1,x_2)} \frac{\partial g_w(x_1,x_2)}{\partial h_w(x_1,x_2)} \frac{\partial h_w(x_1,x_2)}{\partial w_1}$$

$$\begin{aligned}
\frac{\partial J(w)}{\partial g_w(x_1,x_2)} &= -\frac{1}{n}\sum_{i=0}^{n}\left[y^{(i)}\frac{1}{g_w(x_1^{(i)},x_2^{(i)})} + (1-y^{(i)}) \times \frac{1}{1-g_w(x_1^{(i)},x_2^{(i)})} \times (-1)\right] \\
&= -\frac{1}{n}\sum_{i=0}^{n}\left[\frac{y^{(i)}}{g_w(x_1^{(i)},x_2^{(i)})} - \frac{1-y^{(i)}}{1-g_w(x_1^{(i)},x_2^{(i)})}\right] \\
&= -\frac{1}{n}\sum_{i=0}^{n}\left[\frac{y^{(i)}(1-g_w(x_1^{(i)},x_2^{(i)}))}{g_w(x_1^{(i)},x_2^{(i)})(1-g_w(x_1^{(i)},x_2^{(i)}))} - \frac{g_w(x_1^{(i)},x_2^{(i)})(1-y^{(i)})}{g_w(x_1^{(i)},x_2^{(i)})(1-g_w(x_1^{(i)},x_2^{(i)}))}\right] \\
&= -\frac{1}{n}\sum_{i=0}^{n}\left[\frac{y^{(i)} \cancel{- y^{(i)} g_w(x_1^{(i)},x_2^{(i)})} - g_w(x_1^{(i)},x_2^{(i)}) \cancel{+ y^{(i)} g_w(x_1^{(i)},x_2^{(i)})}}{g_w(x_1^{(i)},x_2^{(i)})(1-g_w(x_1^{(i)},x_2^{(i)}))}\right] \\
&= \frac{1}{n}\sum_{i=0}^{n}\left[\frac{-y^{(i)} + g_w(x_1^{(i)},x_2^{(i)})}{g_w(x_1^{(i)},x_2^{(i)})(1-g_w(x_1^{(i)},x_2^{(i)}))}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g_w(x_1,x_2)}{\partial h_w(x_1,x_2)} &= \frac{\partial(1+e^{-h_w(x_1,x_2)})^{-1}}{\partial h_w(x_1,x_2)} = -(1+e^{-h_w(x_1,x_2)})^{-2} \times \frac{\partial(1+e^{-h_w(x_1,x_2)})}{\partial h_w(x_1,x_2)} \\
&= -(1+e^{-h_w(x_1,x_2)})^{-2} \times -e^{-h_w(x_1,x_2)} = \frac{e^{-h_w(x_1,x_2)}}{(1+e^{-h_w(x_1,x_2)})^2} \\
&= \frac{e^{-h_w(x_1,x_2)}}{(1+e^{-h_w(x_1,x_2)})(1+e^{-h_w(x_1,x_2)})} = \frac{1}{(1+e^{-h_w(x_1,x_2)})}\frac{e^{-h_w(x_1,x_2)}}{(1+e^{-h_w(x_1,x_2)})} \\
&= \frac{1}{(1+e^{-h_w(x_1,x_2)})}\frac{e^{-h_w(x_1,x_2)}+1-1}{(1+e^{-h_w(x_1,x_2)})} = \frac{1}{(1+e^{-h_w(x_1,x_2)})}\left(1+\frac{-1}{(1+e^{-h_w(x_1,x_2)})}\right) \\
&= g_w(x_1,x_2)(1-g_w(x_1,x_2))
\end{aligned}$$

$$\frac{\partial h_w(x_1,x_2)}{\partial w_1} = 1$$

Finally:

$$\begin{aligned}
\frac{\partial J(w)}{\partial w_1} &= \frac{1}{n}\sum_{i=0}^{n}\left[\frac{-y^{(i)} + g_w(x_1^{(i)},x_2^{(i)})}{\cancel{g_w(x_1^{(i)},x_2^{(i)})(1-g_w(x_1^{(i)},x_2^{(i)}))}} \times \cancel{g_w(x_1^{(i)},x_2^{(i)})(1-g_w(x_1^{(i)},x_2^{(i)}))}\right] \\
&= \frac{1}{n}\sum_{i=0}^{n}\left[-y^{(i)} + g_w(x_1^{(i)},x_2^{(i)})\right]
\end{aligned}$$

Similarly:

$$\frac{\partial J(w)}{\partial w_2} = \frac{1}{n}\sum_{i=0}^{n} x_1 \left[-y^{(i)} + g_w(x_1^{(i)}, x_2^{(i)})\right]$$
$$\frac{\partial J(w)}{\partial w_1} = \frac{1}{n}\sum_{i=0}^{n} x_2 \left[-y^{(i)} + g_w(x_1^{(i)}, x_2^{(i)})\right]$$

For more informations on binary logistic regression, here are usefull links:

- Logistic Regression – ML Glossary documentation
- Derivative of the Binary Cross Entropy

1 Desision Boundary

The method used here is similar to the one used here. In binary logistic regression, decision boundary is located where:\

$$g_w(x_1, x_2) = 0.5 \implies h_w(x_1, x_2) = 0$$

In addition we now that our decision boundary has the following form

$$x_2 = ax_1 + b$$

Thus, we can easily deduce b since if $x_1 = 0$ we have $x_2 = a \times 0 + b \implies x_2 = b$. Thus:

$$h_w(0, x_2) = w_1 + w_3 x_2 = 0 \implies x_2 = \frac{-w_1}{w_3} \tag{4}$$

To deduce the a coefficient, it is slighly more complicated. If we know two points (x_1^a, x_2^a) and (x_1^b, x_2^b) on the decision boundary line, we know that $a = \frac{x_2^b - x_2^a}{x_1^b - x_1^a}$. thus if we compute:

$$h_w(X_1^b, x_2^b) - h_w(X_1^a, x_2^a) = \cancel{w_1} + w_2 x_1^b + w_3 x_2^b \cancel{- w_1} - w_2 x_1^a - w_3 x_2^a = 0$$
$$\implies w_2(x_1^b - x_1^a) + w_3(x_2^b - x_2^a) = 0 \implies \frac{w_2}{-w_3} = \frac{(x_1^b - x_1^a)}{(x_2^b - x_2^a)} = a$$

Thus we have the decision boundary defined as follow:

$$d(x) = \frac{w_2}{-w_3} x - \frac{w_1}{w_3}$$